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On the Norm Groups of Global Fields

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INTRODUCTION

Let K/k be a finite extension of fields. We define

$$N_{K/k}K^* = \{N_{K/k}(\alpha)/\alpha \in K^*\},$$

where K^* is the multiplicative group of K, and $N_{K/k}(\alpha)$ is the norm of α from K to k. The subgroup $N_{K/k}K^*$ of k^* is called a *norm group* of the field k. One of the fundamental objects of study in local class field theory is the family of norm groups of local fields (i.e., fields complete for a discrete valuation with finite residue field). It is proved in local class field theory that the mapping $K \mapsto N_{K/k}K^*$ from the set of finite Abelian extensions of a local field k into the set of norm groups of k is a bijection. It is also shown that $N_{K/k}K^* = N_{K'/k}K'^*$ for any finite separable extension K of a local field k, where K' is the maximal Abelian extension of k contained in K [12]. In this paper we show that the mapping $K \mapsto N_{K/k}K^*$ from the set of finite extensions of a global field k (i.e., an algebraic number field or a function field in one variable over a finite field) into the set of its norm groups is essentially different from that in the local case. Our result (Theorem 1) says that $K \subseteq L$ iff $N_{L/k}L^* \subseteq N_{K/k}K^*$ for any finite Galois extensions K and L of a global field k. In particular, this implies $K = L$ iff $N_{K/k}K^* = N_{L/k}L^*$.

In connection with our result it is worth mentioning the following result of Neukirch [8]. Let Ω be the algebraic closure of the field of rational numbers $\mathbf{Q}$. Let G_K denote the absolute Galois group of an algebraic number field K, i.e., $G_K = G(\Omega/K)$. Neukirch [8] proved that for any finite Galois extensions K and L of $\mathbf{Q}$, if G_K and G_L are isomorphic as topological groups with the profinite topology, then $K = L$. This result has been subsequently generalized independently by Ikeda, Iwasawa, and Uchida [16]: for any finite extensions K and L of $\mathbf{Q}$, if G_K and G_L are isomorphic as topological groups, then $K \cong L$ (i.e., K and L are conjugate

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over $\mathbf{Q}$). Two other interesting notions of equivalence of algebraic number fields have been investigated in recent years. Two algebraic number fields K and L are *arithmetically equivalent* iff they have identical zeta functions: $\zeta_K(s)=\zeta_L(s)$. Gassmann [4] and more recently Perlis [10] have shown that arithmetically equivalent fields have the same normal closure over $\mathbf{Q}$, but are not necessarily isomorphic. Sonn [13] introduced the following notion of equivalence of algebraic number fields. A finite extension K of a field k is *k-adequate* if there is a finite dimensional k-central division algebra containing K as a maximal subfield. A finite group G is *k-admissible* if there is a Galois extension K/k with $G\cong G(K/k)$, and K is k-adequate. Both notions were introduced by Schacher [11] and investigated by him and others. Sonn [13] proved that if K and L are algebraic number fields such that for every finite group G, G is K-admissible iff G is L-admissible, then K and L have the same normal closure and the same degree over $\mathbf{Q}$.

In this paper we introduce a notion of equivalence of global fields, whose relation to the above equivalence relations is not clear. It is well known that if finite separable extensions K and L of a global field k are conjugate over k (k-isomorphic), then $N_{K/k}K^*=N_{L/k}L^*$. Our result (Theorem 1) shows that the converse is true if K and L are Galois over k. Finite separable extensions K and L of a global field k are *k-equivalent* if $N_{K/k}K^*=N_{L/k}L^*$. We are thus led to the following question: does k-equivalence of K and L imply that K and L are conjugate over k?

Let K and L be finite Galois extensions of a global field k. Let F be the compositum of K and L ($F=KL$). We prove (Theorem 1) that the index of $N_{F/k}F^*$ in $N_{K/k}K^*\cap N_{L/k}L^*$ is finite. By local class field theory the corresponding index is one whenever K and L are finite Abelian extensions of a local field k. An example will be given to show that in general the index is different from one in the global case, even if F is the compositum of Abelian extensions K/k and L/k. The question of whether or not an equality $N_{F/k}F^*=N_{K/k}K^*\cap N_{L/k}L^*$ holds in the global case is shown to be related to the so-called Hasse norm principle (HNP). We say that a finite Galois extension E/k has *Property I* if for any Galois extensions $k\subseteq KL\subseteq E$ of k the equality $N_{F/k}F^*=N_{K/k}K^*\cap N_{L/k}L^*$ holds. By local class field theory every finite Abelian extension E/k of a local field k has Property I. We prove in Theorem 7 that a finite Abelian extension E/k of a global field k has Property I iff HNP holds for E/k. Our result for non-Abelian Galois extensions is Proposition 5. In the second section we apply Theorem 1 to obtain the following result. There is a subfamily of irreducible over a global field k decomposable forms such that $f(x_1, ..., x_n)$ and $g(y_1, ..., y_m)$ are equivalent (over k) if and only if

$$\{f(a_1, ..., a_n)/a_i\in k\}=\{g(b_1, ..., b_m)/b_i\in k\}.$$

(For details see Theorem 8.)

1. The Norm Groups of Global Fields

In this section we will use class field theory. The notation is fairly standard but is summarized below for the convenience of the reader. Let G be a finite group, written multiplicatively. Let A be a G-module, written additively or multiplicatively:

$$A^G = \{a \in A/\sigma a = a \text{ for all } \sigma \in G\}$$

$$N_G = \sum_{\sigma \in G} \sigma \in \mathbf{Z}[G]$$

$$I_G = \text{the ideal in } \mathbf{Z}[G] \text{ generated by } \{\sigma - 1/\sigma \in G\}$$

$$_{N_G}A = \{a \in A/N_G a = 0\}$$

$$N_G A = \{N_G a/a \in A\}.$$

The Tate cohomology is used:

$$\hat{H}^n(G, A) = \begin{cases} H^n(G, A) & \text{if } n \geqslant 1 \\ A^G/N_G A & \text{if } n = 0 \\ _{N_G}A/I_G A & \text{if } n = -1 \\ H_{-n-1}(G, A) & \text{if } n \leqslant -2. \end{cases}$$

Let K/k be a finite Galois extensions of a global field k with the Galois group G. The subgroup

$$N_{K/k} K^* = N_G K^* = \{N_{K/k}(a)/a \in K^*\}$$

of the multiplicative group k^* of k is a *norm group* of k, where $N_{K/k}(a) = N_G a = \prod_{\sigma \in G} \sigma(a)$ for $a \in K^*$. Let J_K and C_K be the idele group and the idele class group of K, respectively. The short exact sequence of G-modules

$$1 \to K^* \to J_K \to C_K \to 1$$

yields the exact sequence of cohomology groups

$$\cdots \longrightarrow \hat{H}^{-1}(G, J_K) \longrightarrow \hat{H}^{-1}(G, C_K) \xrightarrow{g} \hat{H}^0(G, K^*)$$

$$\xrightarrow{f} \hat{H}^0(G, J_K) \longrightarrow \hat{H}^0(G, C_K) \longrightarrow 1.$$

It is well known in global class field theory that the groups $\hat{H}^{-1}(G, C_K)$ and $\hat{H}^{-3}(G, \mathbf{Z})$ are isomorphic. Since $\hat{H}^{-3}(G, \mathbf{Z})$ is dual to the Schur multiplier (of G) $H^3(G, \mathbf{Z}) \cong H^2(G, \mathbf{Q}/\mathbf{Z})$, and the Schur multiplier of G is a finite group, it follows that $\hat{H}^{-1}(G, C_K)$ is finite. Hence the kernel of f, being equal to Im g, is a finite group. The kernel of f has been the subject of investigation in connection with the Hasse norm principle (see [6] and references therein). For each prime v of the field k we fix a k-embedding of the separable closure $\tilde{k}$ of k into the separable closure $\tilde{k}_v$ of the completion k_v of k at v. Note that a k-embedding of $\tilde{k}$ into $\tilde{k}_v$ determines a unique extension of v to $\tilde{k}$. If M is a finite separable extension of k, then the compositum Mk_v is the completion of M at the extension of v to M corresponding to the k-embedding of $\tilde{k}$ into $\tilde{k}_v$. We denote this completion of M by M_v. We will now find the kernel of f. By global class field theory

$$\hat{H}^0(G, J_K) = \coprod_v \hat{H}^0(G(K_v/k_v), K_v^*),$$

where v ranges over all primes of k. Hence the homomophism f is equivalent to the canonical homomorphism

$$k^*/N_{K/k}K^* \to \coprod_v k_v^*/N_{K_v/k_v}K_v^*$$

(v ranges over all primes of k). We set

$$N(K/k) = \bigcap_v (k^* \cap N_{K_v/k_v}K_v^*),$$

where v ranges over the set of primes of k. For the kernel of f we have the following equality

$$\operatorname{Ker} f = N(K/k)/N_{K/k}K^*.$$

Let $i(K/k)$ denote the order of the factor group of $N(K/k)$ by $N_{K/k}K^*$. The Hasse norm theorem asserts that if K/k is a cyclic extension, then $i(K/k) = 1$. We say, more generally, that the Hasse norm principle holds for K/k if $i(K/k) = 1$.

THEOREM 1. *Let K and L be finite Galois extensions of a global field k. Let $F = KL$ be the compositum of K and L. Then*

(a) *$N_{F/k}F^*$ is a subgroup of finite index in $N_{K/k}K^* \cap N_{L/k}L^*$.*

(b) *$K \subseteq L$ iff $N_{L/k}L^* \subseteq N_{K/k}K^*$. In particular, $K = L$ iff $N_{K/k}K^* = N_{L/k}L^*$.*

Proof. Let K and L be finite Galois extensions of k, and $F = KL$. We will show first that $N(F/k)$ is a subgroup of finite index in $N(K/k) \cap N(L/k)$. To show this we need the following two well-known facts in local class field theory. Let v be an arbitrary prime of k. Let M be a finite

Galois extension of k_v, and let M' be the maximal Abelian extension of k_v contained in M. By local class field theory

$$N_{M/k_v} M^* = N_{M'/k_v} M'^*.$$

Moreover, $N_{M/k_v} M^*$ is a subgroup of finite index in the multiplicative group k_v^* of k_v. Suppose P and R are finite Abelian extensions of k_v. Let $T = PR$. By local class field theory

$$N_{T/k_v} T^* = N_{P/k_v} P^* \cap N_{R/k_v} R^*.$$

We will now prove that the index of $N(F/k)$ in $N(K/k) \cap N(L/k)$ is finite. Let S be the set of primes of k that are ramified either in K or in L. K_v and L_v are Abelian extensions of k_v for each $v \notin S$. Hence

$$N_{K_v L_v/k_v}(K_v L_v)^* = N_{K_v/k_v} K_v^* \cap N_{L_v/k_v} L_v^*$$

for each $v \notin S$, where $K_v L_v$ is the compositum of K_v and L_v. Since $F_v = K_v L_v$ for each prime v of k, it follows that

$$\bigcap_{v \notin S} (k^* \cap N_{F_v/k_v} F_v^*) = \bigcap_{v \notin S} (k^* \cap N_{K_v/k_v} K_v^* \cap N_{L_v/k_v} L_v^*). \tag{1}$$

For each $v \in S$, in general, $N_{F_v/k_v} F_v^*$ is a subgroup of finite index in $N_{K_v/k_v} K_v^* \cap N_{L_v/k_v} L_v^*$. Since S is a finite set,

$$\bigcap_{v \in S} (k^* \cap N_{F_v/k_v} F_v^*) \subseteq \bigcap_{v \in S} (k^* \cap N_{K_v/k_v} K_v^* \cap N_{L_v/k_v} L_v^*) \tag{2}$$

is a subgroup of finite index. By (1) and (2)

$$N(F/k) \subseteq N(K/k) \cap N(L/k) \tag{3}$$

is a subgroup of finite index. We have already noted that

$$N_{F/k} F^* \subseteq N(F/k) \tag{4}$$

is a subgroup of finite index $i(F/k)$. On the other hand, we have

$$N_{F/k} F^* \subseteq N_{K/k} K^* \cap N_{L/k} L^* \subseteq N(K/k) \cap N(L/k). \tag{5}$$

Thus by (3)–(5), $N_{F/k} F^*$ is subgroup of finite index in $N_{K/k} K^* \cap N_{L/k} L^*$. This proves the first assertion of the theorem.

Suppose that $k \subseteq X \subseteq Y$ are finite Galois extensions of k. We will show that if $X \neq Y$, then $N_{Y/k} Y^*$ is a subgroup of infinite index in $N_{X/k} X^*$. The short exact sequence

$$1 \to X^* \to J_X \to C_X \to 1$$

yields the exact sequence of cohomology groups

$$\cdots \to \hat{H}^0(G_1, X^*) \to \hat{H}^0(G_1, J_X) \to \hat{H}^0(G_1, C_X) \to 1,$$

where $G_1 = G(X/k)$, and J_X, C_X are the idele group and the idele class group of X, respectively. Since

$$\hat{H}^0(G_1, J_X) = \coprod_v \hat{H}^0(G(X_v/k_v), X_v^*) = \coprod_v k_v^*/N_{X_v/k_v} X_v^*$$

(direct sum over all primes of k), the exact sequence of cohomology groups yields the following exact sequence

$$\cdots \to k^*/N_{X/k} X^* \to \coprod_v k_v^*/N_{X_v/k_v} X_v^* \to C_k/N_{X/k} C_X \to 1.$$

Thus we have the short exact sequence of groups

$$1 \to k^*/N(X/k) \to \coprod_v k_v^*/N_{X_v/k_v} X_v^* \to C_k/N_{X/k} C_X \to 1.$$

Similarly we obtain the short exact sequence corresponding to Y. Consider the commutative diagrams of canonical group homomorphisms

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & N(X/k)/N(Y/k) \arrow[r] \arrow[d] & k^*/N(Y/k) \arrow[r] \arrow[d] & k^*/N(X/k) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \coprod_v N_{X_v/k_v} X_v^*/N_{Y_v/k_v} Y_v^* \arrow[r] \arrow[d] & \coprod_v k_v^*/N_{Y_v/k_v} Y_v^* \arrow[r] \arrow[d] & \coprod_v k_v^*/N_{X_v/k_v} X_v^* \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N_{X/k} C_X/N_{Y/k} C_Y \arrow[r] \arrow[d] & C_k/N_{Y/k} C_Y \arrow[r] \arrow[d] & C_k/N_{X/k} C_X \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

The rows and the right two columns of the above diagram are short exact sequences. Hence the left column is also a short exact sequence. By global class field theory $N_{Y/k} C_Y$ is a subgroup of finite index in $N_{X/k} C_X$. If X is a proper subfield of Y, then by Theorem 2 [1, p. 12] there exist an infinite number of primes v of k that are unramified in Y and such that X_v is a proper subfield of Y_v. It follows that the middle group in the left column is infinite. Thus $N(X/k)/N(Y/k)$ is an infinite group whenever X is a proper subfield of Y and both are finite Galois extensions of k. Since both groups $N(X/k)/N_{X/k} X^*$ and $N(Y/k)/N_{Y/k} Y^*$ are finite, it follows that $N_{Y/k} Y^*$ is a subgroup of infinite index in $N_{X/k} X^*$ whenever X is a proper subfield of Y and both X and Y are Galois over k.

To prove (b) of the theorem we assume that $N_{L/k}L^* \subseteq N_{K/k}K^*$. By (a) $N_{F/k}F^*$ $(F=KL)$ is a subgroup of finite index in $N_{K/k}K^* \cap N_{L/k}L^* = N_{L/k}L^*$. This is possible, as we have just established, only if $F=L$. Since F is the compositum of K and L, it follows that $K \subseteq L$. The opposite direction is obvious. ∎

Remark 1. A comparison between our result (Theorem 1(b)) and Bauer's theorem [8, p. 310] suggests a possibility of an alternative proof of Theorem 1(b). Later an alternative proof (of Theorem 1) using Bauer's theorem was given by J. Sonn in the algebraic number field case. In the process of proving Theorem 1 we have shown that $N_{Y/k}Y^*$ is a subgroup of infinite index in $N_{X/k}X^*$, whenever X is a proper subfield of Y, and both X, Y are finite Galois extensions of a global field k. This implies a result of R. Klein which was mentioned in [7, p. 157].

Let K and L be finite Galois extensions of a global field k. Let F be the compositum of K and L. We set

$$i_g(k, K, L) = [N_{K/k}K^* \cap N_{L/k}L^* : N_{F/k}F^*]$$

$$i_l(k, K, L) = [N(K/k) \cap N(L/k) : N(F/k)]$$

$$i_h(k, K, L) = [N(K/k) \cap N(L/k) : N_{K/k}K^* \cap N_{L/k}L^*].$$

The indices defined above are finite by (3)–(5) in the proof of Theorem 1. A relation between these indices and $i(F/k)$ is given in the following proposition. (The definition of $i(F/k)$ is given before Theorem 1.)

PROPOSITION 2. *Let K and L be finite Galois extensions of a global field k. Let $F=KL$. Then*

$$i_g(k, K, L) \cdot i_h(k, K, L) = i(F/k) \cdot i_l(k, K, L).$$

Moreover, if K and L are locally Abelian extensions of k, then

$$i(F/k) = i_g(k, K, L) \cdot i_h(k, K, L).$$

Proof. The following diagram shows that the first equality of Proposition 2 holds:

$$\begin{tikzcd}[column sep=small]
 & N(K/k) \cap N(L/k) \arrow[dl, dash] \arrow[dr, dash] & \\
N_{K/k}K^* \cap N_{L/k}L^* \arrow[dr, dash] & & N(F/k) \arrow[dl, dash] \\
 & N_{F/k}F^* &
\end{tikzcd}$$

If K and L are locally Abelian extensions of k, then (2) in the proof of Theorem 1 becomes an equality. Hence (3) in this case is an equality. This means that $i_l(k, K, L) = 1$. ∎

We have already noted that for any finite Abelian extensions K and L of a local field k

$$N_{F/k}F^* = N_{K/k}K^* \cap N_{L/k}L^*,$$

where $F = KL$. The following example shows that in the case of global fields a similar result, in general, does not hold.

EXAMPLE 1. We consider here Scholz's classical biquadratic counter example to the Hasse norm principle [3, p. 199]. Set $K = \mathbf{Q}(\sqrt{13})$, $L = \mathbf{Q}(\sqrt{17})$, and $F = \mathbf{Q}(\sqrt{13}, \sqrt{17})$. By the Hasse norm theorem $i(K/\mathbf{Q}) = i(L/\mathbf{Q}) = 1$. Hence $i_h(\mathbf{Q}, K, L) = 1$. By Proposition 2, $i(F/\mathbf{Q}) = i_g(\mathbf{Q}, K, L)$. Since $i(F/\mathbf{Q}) = 2$ [3, p. 199], it follows that $N_{F/\mathbf{Q}}F^*$ is a subgroup of index 2 in $N_{K/\mathbf{Q}}K^* \cap N_{L/\mathbf{Q}}L^*$. It is shown in [3, p. 360] that 5^2 is a local norm everywhere from F but is not a global norm. Since $N_{K/\mathbf{Q}}K^* \cap N_{L/\mathbf{Q}}L^* = N(F/\mathbf{Q})$, it follows that the factor group of $N_{K/\mathbf{Q}}K^* \cap N_{L/\mathbf{Q}}L^*$ by $N_{F/\mathbf{Q}}F^*$ is generated by the coset with the representative 5^2.

The following corollary is an immediate consequence of Proposition 2.

COROLLARY 3. *Let K and L be finite Abelian extensions of a global field k. Let $F = KL$. If HNP holds for F/k, then*

$$N_{F/k}F^* = N_{K/k}K^* \cap N_{L/k}L^*.$$

The converse of Corollary 3, however, is false as is sown in the following example.

EXAMPLE 2. Let $K = \mathbf{Q}(\sqrt{7})$, $L = \mathbf{Q}(\sqrt{p}, \sqrt{q})$, where $p = 43$, $q = 1721$ are prime numbers. Let $F = KL$. By Proposition 5 [6, p. 252], $i(F/\mathbf{Q}) = 2$, since $7 \equiv -1 \pmod 4$, $p \equiv 1 \pmod 7$, $p \equiv 7 \pmod 8$, $q \equiv 1 \pmod{8p}$, and $(q/7) = -1$. In order to prove that

$$N_{F/\mathbf{Q}}F^* = N_{K/\mathbf{Q}}K^* \cap N_{L/\mathbf{Q}}L^*,$$

it suffices, by Proposition 2, to show that $i_h(\mathbf{Q}, K, L) \neq 1$. Let S be the set of primes of $\mathbf{Q}$ that split in $\mathbf{Q}(\sqrt{p})$. Consider the group homomorphism $\varphi\colon \mathbf{Q}^* \to \{-1, 1\}$ defined by

$$\varphi(x) = \prod_{l \in S} (q, x)_l,$$

where $(\, , \,)_l$ is the quadratic norm residue symbol. We have

$$\varphi(x) = \left(\frac{x'}{q}\right) \cdot \prod_{l \in S \setminus \{q\}} \left(\frac{q}{l}\right)^{v_l(x)},$$

where $v_l(x)$ is the order of x at l, and $x = q^{v_q(x)} \cdot x'$. A prime $t \notin \{2, 43\}$ of $\mathbf{Q}$ splits in $\mathbf{Q}(\sqrt{p})$ if and only if $(p/t) = 1$. Hence if a prime t of $\mathbf{Q}$ is such that $(p/t) = -1$, then $\varphi(t) = (t/q)$. 2, p, q are the only primes of $\mathbf{Q}$ that ramify in L. Since the local degrees of $L/\mathbf{Q}$ at the primes 2, p, q are equal to 2, it follows that all local degrees of $L/\mathbf{Q}$ are 1 or 2. So by [3, p. 360], $\operatorname{Ker} \varphi = \{x \in \mathbf{Q}^* / x^2 \in N_{L/\mathbf{Q}} L^*\}$. Since $(\frac{43}{23}) = (\frac{23}{1721}) = -1$, $23 \notin \operatorname{Ker} \varphi$, thus, $23^2 \notin N_{L/\mathbf{Q}} L^*$. On the other hand, $23^2 \in N(K/\mathbf{Q}) \cap N(L/\mathbf{Q})$.

PROPOSITION 4. *Let F be the compositum of finite cyclic extensions K and L of a global field k. Let $n = (F{:}k)$ and let n_0 be the least common multiple of the local degrees of F/k. Then*

(i) $N(F/k) = N_{K/k} K^* \cap N_{L/k} L^*$

(ii) $N(F/k)/N_{F/k} F^*$ *is cyclic of order* n/n_0.

Proof. Since K and L are cyclic extensions of k, $N(K/k) = N_{K/k} K^*$ and $N(L/k) = N_{L/k} L^*$ by the Hasse norm theorem. This means that $i_h(k, K, L) = 1$. On the other hand, $i_l(k, K, L) = 1$, since F/k is an Abelian extension. So $N(F/k) = N_{K/k} K^* \cap N_{L/k} L^*$. Let G be the Galois group of the extension F/k. G is isomorphic to a subgroup of $G(K/k) \times G(L/k)$. So G itself is a direct product of two cyclic groups: $G = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$. Let $k \subseteq T_j \subseteq F$ be the field corresponding to $\langle \sigma_j \rangle$ $(j = 1, 2)$. Since F is the compositum of cyclic extensions T_1 and T_2 of k, $N(F/k) = N_{T_1/k} T_1^* \cap N_{T_2/k} T_2^*$ by (i). By Theorem 10 [15, p. 30]

$$H^3(G, F^*) \cong N_{T_1/k} T_1^* \cap N_{T_2/k} T_2^* / N_{F/k} F^*.$$

$H^3(G, F^*)$ is the cyclic group of order n/n_0 [3, p. 199]. Thus $N(F/k)/N_{F/k} F^*$ is cyclic of order n/n_0. ▮

Remark 2. Let F be a compositum of two cyclic extensions of a global field k. Let $G = G(F/k)$, and let G_p be the Sylow p-subgroup of G. By Proposition 4, HNP holds for F/k iff for each prime p dividing the order of G there is a prime v of k such that $G_p \subseteq G(F_v/k_v)$. In particular, if G is a p-group we recover Corollary 5.3 [5, p. 24]. Although the above result on HNP for F/k follows by Corollaries 2.4 and 5.3 [5], Proposition 4 is of interest because it determines the "number knot" of F/k [6], i.e., the factor group $N(F/k)/N_{F/k} F^*$.

We say that a finite Galois extension E/k has *Property I*, if for any Galois extensions $k \subseteq KL \subseteq E$ of k the equality

$$N_{F/k}F^* = N_{K/k}K^* \cap N_{L/k}L^*$$

holds, where $F = KL$. By local class field theory every finite Abelian extension E of a local field k has Property I. In this connection it is natural to pose the following question: What are the Galois extensions of global fields that have Property I? We will need the following notation. Let F be the compositum of finite Galois extensions K and L of a global field k. Let v be a prime of k. We set

$$N(k, K, L) = N(K/k) \cap N(L/k)/N(F/k),$$

$$N_v(k, K, L) = N_{K_v/k_v}K_v^* \cap N_{L_v/k_v}L_v^*/N_{F_v/k_v}F_v^*.$$

Let $G = G(F/k)$, $H = G(F/K)$, and $N = G(F/L)$. We denote by G_v, H_v, and N_v the Galois groups of the corresponding local extensions at v. We set

$$G(k, K, L) = G'H \cap G'N/G' \qquad \text{and} \qquad G_v(k, K, L) = G'_vH_v \cap G'_vN_v/G'_v,$$

where G' and G'_v are the commutator subgroups of G and G_v, respectively.

Proposition 5. *Let k be a global field.*

(a) *Let K and L be finite Galois extensions of k. Then the following sequence of canonical group homomorphisms*

$$1 \to N(k, K, L) \to \coprod_v N_v(k, K, L) \to G(k, K, L) \tag{6}$$

(*direct sum over all primes of k*) *is an exact sequence.*

(b) *Let E be a finite Galois extension of k. Suppose that HNP holds for every Galois extension $k \subseteq T \subseteq E$ of k. Then E/k has Property I iff for any Galois extensions $k \subseteq KL \subseteq E$ of k the canonical homomorphism*

$$\coprod_v G_v(k, K, L) \to G(k, K, L) \tag{7}$$

(*direct sum over all primes of k*) *is a monomorphism.*

Proof. Let F be the compositum of finite Galois extensions K and L of k. The exact sequence of cohomology groups

$$\cdots \to \hat{H}^0(G, F^*) \to \hat{H}^0(G, J_F) \to \hat{H}^0(G, C_F) \to 1$$

$(G=G(F/k))$ yields the short exact sequence

$$1 \to k^*/N(F/k) \to \coprod_v k_v^*/N_{F_v/k_v} F_v^* \to C_k/N_{F/k} C_F \to 1.$$

Let F' be the maximal Abelian extension of k contained in F. We denote by Ψ_F the composition of the group homomorphisms

$$\coprod_v k_v^*/N_{F_v/k_v} F_v^* \twoheadrightarrow C_k/N_{F/k} C_F \rightarrowtail\!\!\!\twoheadrightarrow G(F'/k),$$

where the second isomorphism is the reciprocity isomorphism. Let $k \subseteq M \subseteq F'$ be the fixed field of $\Psi_F[\coprod_v N_v(k, K, L)]$. Consider the commutative diagrams of canonical group homomorphisms

$$\begin{tikzcd}
1 \arrow[r] & N(k, K, L) \arrow[r] \arrow[d, tail] & \coprod_v N_v(k, K, L) \arrow[r, "\Psi_F"] \arrow[d, tail] & G(F'/M) \arrow[r] \arrow[d, tail] & 1 \\
1 \arrow[r] & k^*/N(F/k) \arrow[r] & \coprod_v k_v^*/N_{F_v/k_v} F_v^* \arrow[r, "\Psi_F"] & G(F'/k) \arrow[r] & 1
\end{tikzcd}$$

Since the second row is a short exact sequence, it follows that the first row is also a short exact sequence. To complete the proof of the first assertion of the proposition, it suffices to show that $G(F'/M)$ is a subgroup of $G(F'/K'L') \cong G(k, K, L)$, where K' and L' are the maximal Abelian extensions of k contained in K and L, respectively. Using one of the universal properties of the global norm residue symbols we obtain the commutative diagram

$$\begin{tikzcd}
\coprod_v k_v^*/N_{F_v/k_v} F_v^* \arrow[r, "\Psi_F"] \arrow[d, "\pi"] & G(F'/k) \arrow[d, "\mathrm{res}"] \\
\coprod_v k_v^*/N_{K_v/k_v} K_v^* \arrow[r, "\Psi_K"] & G(K'/k)
\end{tikzcd}$$

where the definition of Ψ_K is similar to that of Ψ_F. Let $\sigma \in G(F'/M)$ be an arbitrary element. Let $a \in \coprod_v N_v(k, K, L)$ be such that $\Psi_F(a) = \sigma$. Since $\pi(a) = 1$, it follows that $\mathrm{res}(\sigma) = 1$. So $\sigma \in G(F'/K')$. Thus K' is contained in M. Similarly we obtain that $L' \subseteq M$. So the compositum $K'L'$ is contained in M.

By global class field theory, the norm residue symbol corresponding to a finite Abelian extension of k is equal to the product of the norm residue symbols corresponding to the local extensions [9, Satz 6.15, p. 260]. Using this theorem we obtain the following commutative diagram

$$\begin{tikzcd} \coprod_v N_v(k, K, L) \arrow[r, "\Psi_F"] \arrow[d, "\rho"'] & G(k, K, L) \\ \coprod_v G_v(k, K, L) \arrow[ur] & \end{tikzcd}$$

where the isomorphism ρ is induced by the product of the local norm residue symbols: $\Pi_v(\ , F_v/k_v)$ (product over all primes of k). Thus the exact sequence (6) is equivalent to the exact sequence

$$1 \to N(k, K, L) \to \coprod_v G_v(k, K, L) \to G(k, K, L). \tag{8}$$

To prove (b) of the proposition we assume that HNP holds for every Galois extension $k \subseteq T \subseteq E$ of k. By Proposition 2, $i_g(k, K, L) = i_l(k, K, L)$ for any Galois extensions $k \subseteq KL \subseteq E$ of k. So E/k has Property I iff $N(k, K, L) = 1$ for any Galois extensions $k \subseteq KL \subseteq E$ of k. Thus by (8), E/k has Property I iff the canonical homomorphism (7) is a monomorphism for any Galois extensions $k \subseteq KL \subseteq E$ of k. ∎

A finite Galois extension E of a global field k is *k-adequate* if there is a finite dimensional k-central division algebra containing E as a maximal subfield [11]. By Proposition 2.6 [11, p. 454], E is k-adequate iff for each prime number $p \mid (E:k)$ there are at least two primes v of k for which $G(E_v/k_v)$ contains a Sylow p-subgroup of $G(E/k)$. Adequate extensions of global fields were investigated by Schacher and others (see [13, 14] and the references therein). We will say that a finite Galois extension E of a global field k is *weakly k-adequate* if for each prime number $p \mid (E:k)$ there is at least one prime v of k for which $G(E_v/k_v)$ contains a Sylow p-subgroup of $G(E/k)$. So every Galois k-adequate extension is weakly k-adequate. The converse is false: $k = \mathbf{Q}$, $E = \mathbf{Q}(\xi)$, where ξ is a primitive 8th root of unity [11, p. 454]. By Theorem 2.5 [5, p. 18], HNP holds for every weakly k-adequate extension. Moreover, if E is weakly k-adequate, then every Galois extension $k \subseteq T \subseteq E$ of k is weakly k-adequate. Thus Proposition 5 yields the following corollary.

COROLLARY 6. *Let k be a global field. If a finite Galois extension E/k is weakly k-adequate, then E/k has Propery I iff the canonical homomorphism*

$$\coprod G_v(k, K, L) \to G(k, K, L)$$

(direct sum over all primes of k) is a monomorphism for any Galois extensions $k \subseteq KL \subseteq E$ of k.

THEOREM 7. *Let E/k be a finite Abelian extension of a global field k. Then E/k has Property I iff HNP holds for E/k.*

Proof. Let $E = L_1 L_2 \cdots L_n$ be the compositum of cyclic extensions L_i of k, $i = 1, 2, \ldots, n$. By local class field theory

$$\begin{aligned}\bigcap_{i=1}^{n} N(L_i/k) &= \bigcap_{i=1}^{n}\left[\bigcap_{v}(k^* \cap N_{L_{iv}/k_v} L_{iv}^*)\right]\\ &= \bigcap_{v}\left[k^* \cap \left(\bigcap_{i=1}^{n} N_{L_{iv}/k_v} L_{iv}^*\right)\right]\\ &= \bigcap_{v}\,[k^* \cap N_{E_v/k_v} E_v^*] = N(E/k).\end{aligned}$$

By the Hasse norm theorem $N(L_i/k) = N_{L_i/k} L_i^*$ for each $i = 1, 2, \ldots, n$. So, if E/k has Property I, then

$$N(E/k) = \bigcap_{i=1}^{n} N(L_i/k) = \bigcap_{i=1}^{n} N_{L_i/k} L_i^* = N_{E/k} E^*.$$

This means NHP holds for E/k. Conversely, suppose NHP holds for E/k. Since E/k is an Abelian extension, HNP holds for any extension $k \subseteq T \subseteq E$ of k [6, p. 226]. E_v/k_v is Abelian for each prime v of k. So $G_v(k, K, L) = 1$ for any fields $k \subseteq KL \subseteq E$ and for each prime v of k. Thus, by Proposition 5, E/k has Property I. ∎

In [11], Schacher proved that for any finite group G there is a Galois extension E/k of an algebraic number field k with $G \cong G(E/k)$ and such that E is k-adequate (Theorem 9.1, p. 472). We use this theorem to show that the following statement is false. "Let E/k be a finite Galois extension of an algebraic number field k. If HNP holds for every Galois extension $k \subseteq T \subseteq E$ of k, then E/k has Property I." The following example of a group G has been communicated to us by M. Schacher: $G = Q_8 \times \langle c \rangle$, where $Q_8 = \langle a, b / a^4 = 1, a^2 = b^2, ab = ba^{-1} \rangle$, and $\langle c \rangle$ is the cyclic group of order two. Let G' be the commutator subgroup of G. Let $N_1 = \langle (1, c) \rangle$, $N_2 = \langle (a^2, c) \rangle$. Then the factor group of $G'N_1 \cap G'N_2 = G'N_1$ by G' is cyclic of order two. By Schacher's theorem there is a Galois k-adequate extension E of an algebraic number field k with $G \cong G(E/k)$. Let v_1, v_2 be two distinct primes of k such that $G \cong G(E_v/k_v)$ for each $v \in \{v_1, v_2\}$. If $k \subseteq KL \subseteq E$ are the fixed fields of N_1 and N_2, respectively, then $G_v(k, K, L)$

is cyclic of order two for each $v \in \{v_1, v_2\}$. Since $G(k, K, L)$ is cyclic of order two, it follows by Corollary 6 that E/k does not have Property I.

2. On the k-Equivalence of Decomposable Forms

Let k be a global field. The form $f(x_1, ..., x_m) \in k[x_1, ..., x_m]$ is called *decomposable* if it factors into linear factors in some separable extension Ω/k. Let K/k be a finite separable extension. Let $\sigma_1, ..., \sigma_n$ be the set of distinct k-embeddings of K into its normal closure over k. For any $\alpha_1, ..., \alpha_m \in K^*$ we set

$$N_{K/k}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \prod_{i=1}^{n} [\sigma_i(\alpha_1) x_1 + \cdots + \sigma_i(\alpha_m) x_m].$$

$N_{K/k}(\alpha_1 x_1 + \cdots + \alpha_m x_m)$ is a decomposable form with coefficients in k. Two forms of the same degree with coefficients in k are called *k-equivalent* if each can be obtained from the other by a linear change of variables with coefficients in k. Using an argument analogous to the proof of Theorem 2 [2, p. 80] we obtain the following characterization of irreducible decomposable forms. If $K = k(\alpha_2, ..., \alpha_m)$ is a finite separable extension of k, then the form

$$N_{K/k}(x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m) \tag{9}$$

is irreducible over k. Every irreducible decomposable form with coefficients in k is k-equivalent to a constant multiple of a form of the type (9).

Let $f(x_1, ..., x_m)$ be a form with coefficients in k. We define $n(f)$ to be the least number of variables in a form that is k-equivalent to $f(x_1, ..., x_m)$. If $f(x_1, ..., x_m)$ is an irreducible (over k) decomposable form, then $n(f) \leqslant \deg f$. Indeed, by the above characterization of irreducible decomposable forms, it suffices to prove the inequality for $f(x_1, ..., x_m)$ of the type (9). If $1, \alpha_2, ..., \alpha_m$ are linearly dependent over k, then without loss of generality we may assume that $\alpha_m = a_1 \cdot 1 + \cdots + a_{m-1}\alpha_{m-1}$. The forms $N_{K/k}(x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m)$ and $N_{K/k}(x'_1 + \alpha_2 x'_2 + \cdots + \alpha_{m-1} x'_{m-1})$ are k-equivalent, since the linear substitutions

$$\begin{array}{ll} x'_1 = x_1 + a_1 x_m, & x_1 = x'_1 \\ \cdots & \cdots \\ x'_{m-1} = x_{m-1} + a_{m-1} x_m, & x_{m-1} = x'_{m-1} \\ & x_m = 0 \end{array}$$

take one into the other. Continuing in this pattern we obtain that

$N_{K/k}(x_1+\alpha_2x_2+\cdots+\alpha_mx_m)$ is k-equivalent to $N_{K/k}(y_1+\beta_2y_2+\cdots+\beta_sy_s)$ with $1, \beta_2, ..., \beta_s$ linearly independent over k. So $s\leqslant(K:k)$. Since $(K:k)$ is equal to the degree of $N_{K/k}(x_1+\alpha_2x_2+\cdots+\alpha_mx_m)$, it follows that $n(f)\leqslant \deg f$.

An irreducible over k decomposable form $f(x_1, ..., x_n)$ is called *k-full* if $n(f)=\deg f$. So every k-full form $f(x_1, ..., x_n)$ is k-equivalent to a constant multiple of a form of the type (9) with $\{1, \alpha_2, ..., \alpha_m\}$ a basis of K/k. We will refer to K as a field corresponding to $f(x_1, ..., x_n)$.

THEOREM 8. *Let k be a global field. Let K and L be fields corresponding to k-full forms $f(x_1, ..., x_n)$ and $g(y_1, ..., y_m)$, respectively. If K/k and L/k are Galois extensions, then f is k-equivalent to g iff*

$$\{f(a_1, ..., a_n)/a_i\in k\}=\{g(b_1, ..., b_m)/b_i\in k\}. \tag{10}$$

Proof. If f and g are k-equivalent, then by the definition of the k-equivalence relation, the equality (10) holds. Conversely, suppose that the equality (10) holds. We may assume without loss of generality that

$$f(x_1, ..., x_n)=aN_{K/k}(x_1+\alpha_2x_2+\cdots+\alpha_nx_n)$$

and

$$g(y_1, ..., y_m)=bN_{L/k}(y_1+\beta_2y_2+\cdots+\beta_my_m),$$

where $a, b\in k^*$, and $\{1, \alpha_2, ..., \alpha_n\}$, $\{1, \beta_2, ..., \beta_m\}$ are bases of K/k and L/k, respetively. Then the equality (10) implies $aN_{K/k}K^*=bN_{L/k}L^*$. So $b^{-1}a\in N_{L/k}L^*$. $N_{L/k}L^*$ is a subgroup of k^*, so $a^{-1}b\in N_{L/k}L^*$. We thus have

$$N_{L/k}L^*=a^{-1}bN_{L/k}L^*=a^{-1}(aN_{K/k}K^*)=N_{K/k}K^*.$$

By Theorem 1(b) we obtain therefore that $K=L$. In particular, $n=m$, and $\{1, \alpha_2, ..., \alpha_n\}$, $\{1, \beta_2, ..., \beta_n\}$ are bases of K/k. Let $\gamma\in K^*$ be such that $a^{-1}b=N_{K/k}(\gamma)$. We have two bases of K/k: $\{\alpha_1=1, \alpha_2, ..., \alpha_n\}$ and $\{\gamma_1=\gamma, \gamma_2=\gamma\beta_2, ..., \gamma_n=\gamma\beta_n\}$. Let

$$\alpha_i=\sum_{j=1}^{n} a_{ij}\gamma_j \qquad (a_{ij}\in k; i=1, ..., n)$$

$$\gamma_i=\sum_{j=1}^{n} c_{ij}\alpha_j \qquad (c_{ij}\in k; i=1, ..., n).$$

The forms f and g are k-equivalent, since the linear substitutions

$$x_i=\sum_{j=1}^{n} c_{ji}\,y_j \qquad (i=1, ..., n)$$

and

$$y_i = \sum_{j=1}^{n} a_{ji} x_j \qquad (i = 1, ..., n)$$

take one into the other. ∎

The following corollary is probably well known, but a short proof using Theorem 8 is included here.

COROLLARY 9. *Let k be a global field of characteristic* $\neq 2$. *Let f and g be nonsingular binary quadratic forms with coefficients in k. Then f is k-equivalent to g iff*

$$\{f(a_1, a_2)/a_i \in k\} = \{g(b_1, b_2)/b_i \in k\}. \tag{11}$$

Proof. Suppose that f is k-equivalent to

$$ax^2 + by^2 = a(x + \sqrt{c}y)(x - \sqrt{c}y),$$

where $c = -b/a$. If f does not represent zero in k, then c is not a square in k. Since factorization in polynomial rings is unique, it follows that f is irreducible over k. So, if f does not represent zero in k, then f is k-full, and $\{f(a_1, a_2)/a_i \in k\}$ is a proper subset of k. If, however, f represents zero in k, then by Theorem 5 [2, p. 393]

$$\{f(a_1, a_2)/a_i \in k\} = k.$$

If f and g are k-equivalent, then the equality (11) holds by the definition of the k-equivalence relation. Conversely, suppose that the equalty (11) holds. If f represents zero, then g represents zero. By Theorem 9 [2, p. 395], f and g are k-equivalent. If f does not represent zero, then both f and g are k-full. By Theorem 8, f and g are k-equivalent. ∎

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